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# Polynomials Orthogonal with Respect to Discrete Convolution

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The concept of "Discrete Convolution Orthogonality" is introduced and investigated. This leads to new orthogonality relations for the Charlier and Meixner polynomials. This in turn leads to bilinear representations for them. We also show that the zeros of a family of convolution orthogonal polynomials are real and simple. This proves that the zeros of the Rice polynomials are real and simple.

## 1. INTRODUCTION

Let $\{\alpha_n, n = 0, 1, 2,...\}$ be a sequence of real or complex numbers with the property that $\alpha_0 = 1$ and $\alpha_n \neq 0$ for all $n \geqslant 1$. For any such sequence the present authors [2] introduced and studied the sequence-to-function transform $L[f; \alpha, x]$ defined by means of

$$L[f; \alpha, x] = \sum_{n=0}^{\infty} (-x)^n \alpha_n \cdot \Delta^n f(0) \tag{1.1}$$

where $\Delta f(x) = f(x + 1) - f(x)$, $\Delta^n f(x) = \Delta \cdot \Delta^{n-1} f(x)$. This transform clearly maps the set of sequences onto the set of formal power series. If the function $\phi(x)$ is the generating function for the given sequence $\{\alpha_n\}$, i.e., $\phi(x) = \sum_{r=0}^{\infty} \alpha_r x^r$, then (1.1) may be rewritten, at least formally, as

$$L[f; \alpha; x] = L[f; \phi; x] = \sum_{n=0}^{\infty} ((-x)^n/n!)\, \phi^{(n)}(x)\, f(n). \tag{1.2}$$

We shall call the mapping defined by (1.1) or (1.2) the $(L, \alpha)$ or $(L, \phi)$ transform and use either of the two notations in the left-hand side of (1.2).

We also proved that

$$L[f * g; \alpha; x] = L[f; \alpha; x] L[g; \alpha; x] \tag{1.3}$$

where $f * g$ is a convolution product of the two sequences $\{f(n)\}$ and $\{g(n)\}$ defined as the sequence whose $n$th component is given by

$$(f * g)(n) = \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{r} \frac{\alpha_i \alpha_{r-i}}{\alpha_r} \Delta^i f(0) \Delta^{r-i} g(0).$$

In particular, if $f(n) = ((-1)^j/\alpha_j)\binom{n}{j}$ (where the nonnegative integer $j$ is fixed) then

$$L\left[\frac{(-1)^j}{a_j}\binom{n}{j}; \alpha; x\right] = x^j, \tag{1.4}$$

so that if $P(n)$ is any polynomial in the discrete variable $n$ then

$$L[P; \alpha; x] = P^*(x),$$

where $P^*$ is a polynomial (of the same degree as $P$) in the continuous variable $x$. Using this fact we introduce here a discrete analog of the concept of "Orthogonality with respect to convolution" which was introduced in [3]. We shall see that there are polynomial sets which are not necessarily orthogonal in the "ordinary sense", i.e., with respect to the inner product

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) \, d\psi(x), \tag{1.5}$$

where $\psi(x)$ is an increasing function which is of bounded variation in $(-\infty, \infty)$ with infinitely many points of increase and finite moments of all orders (see [14, p. 18]) but are nevertheless orthogonal with respect to another inner product. We shall find that there are polynomial sets which are orthogonal in this sense of (1.5) and also with respect to a discrete convolution. This gives new orthogonality relations, connection relations as well as several bilinear generating functions for these polynomials. The Charlier and the Meixner polynomials form such sets.

We shall also see that if the function $\phi(x)$ of (1.2) is a Laplace transform of a nonnegative function, then the corresponding convolution orthogonal polynomials will have real and simple zeros. An example is the set of Rice polynomials $H_n(\xi, p, v)$ for $p > 0$ and $v < 0$.

The transformation (1.2) first appeared in summability theory. It was introduced by Jakimovski [10] and called a $J$-transform. The $[J, f(x)]$ limit of a sequence $\{g(n)\}_0^\infty$ is $\lim_{x\to\infty} \sum_{n=0}^{\infty} ((-x)^n/n!) f^{(n)}(x) g(n)$. Jakimovski [10] characterized the regular $[J, f(x)]$ transforms. Later, Leviatan and Lorch [11] showed that it is totally regular if and only if $f(x)$ is completely monotonic. The $[J, f(x)]$ transformations obviously contain the Borel transform, the Abel transform and the $A^{(\alpha)}$ transforms of Borwein [4] as the special cases $f(x) = e^{-x}$, $(1 + x)^{-1}$, and $(1 + x)^{-\alpha-1}$ $(\alpha \geqslant 0)$, respectively. Recently, Ismail [9] extended the $[J, f(x)]$ means to double sequences and characterized the resulting regular and totally regular means.

In the present work we treat only one-dimensional transforms. However, since we are using power series, we can interchange the order of summation and everything carries over to higher dimensions. The two-dimensional $(L, \phi)$ transform is

$$L[f; \phi; x, y] = \sum_{m,n=0}^{\infty} \frac{(-x)^m}{m!} \frac{(-y)^n}{n!} \frac{\partial^{m+n}\phi(x, y)}{\partial x^m \partial y^m} f(m, n).$$

The difficulty in summability is indeed that we cannot always interchange the summations.

We have also obtained basic analogs of the transformation (1.2). This is somewhat unrelated to the present work and will be treated elsewhere. However, one can trivially obtain basic analogs of the present paper.

## 2. Discrete Convolution Orthogonality

Let $\{P_j(x); j = 0, 1, 2,...\}$ be a polynomial set which is orthogonal with respect to (1.5), i.e.,

$$\int_{-\infty}^{\infty} P_j(x) P_l(x)\, d\psi(x) = \lambda_j \delta_{lj}, \tag{2.1}$$

where $\lambda_j > 0$ and $\delta_{lj}$ is the familiar Krönecker delta.

In view of (1.4), for each nonnegative integer $j$, there is a polynomial $Q_j(n)$ of degree $j$ in $n$ $(n = 0, 1,...)$ such that

$$L[Q_j; \alpha; x] = P_j(x).$$

From (1.3) we see, further, that (2.1) is equivalent to

$$\int_{-\infty}^{\infty} L[Q_j * Q_l; \alpha; x]\, d\psi(x) = \lambda_j \delta_{lj}, \qquad \lambda_j > 0. \tag{2.2}$$

Relation (2.2) expresses what we call discrete convolution orthogonality. It can also be expressed formally as

$$\sum_{m,n=0}^{\infty} \mu_{m,n} Q_j(n) Q_l(m) = \lambda_j \delta_{lj}, \tag{2.3}$$

where

$$\mu_{m,n} = \int_{-\infty}^{\infty} (-x)^{m+n} \frac{\phi^{(n)}(x)\,\phi^{(m)}(x)}{n!m!}\, d\psi(x). \tag{2.4}$$

Hence if $\psi(x)$, $\phi(x)$ are given so that (2.4) exists for all $m, n \geqslant 0$ then we can define an inner product

$$(f, g)_c = \sum_{m,n=0}^{\infty} \mu_{m,n} f(n)\, g(m), \tag{2.5}$$

which is positive definite on the space of all polynomials, i.e., $(f, f) \geqslant 0$ and $(f, f) = 0$ if and only if $f = 0$.

In the important special case $\phi(x) = (1 - x)^{-p}$, $p \neq -1, -2,\ldots$, and $(0, 1/v)$, $v > 1$, is the support of $d\psi(x)$, we have

$$\mu_{m,n} = \frac{(p)_n\,(p)_m\,(-1)^{m+n}}{n!m!} \int_0^{1/v} \left(\frac{x}{1-x}\right)^{m+n} \frac{d\psi(x)}{(1-x)^{2p}},$$

where $(p)_n = p(p+1)\cdots(p+n-1)$, $(p)_0 = 1$.

It is obvious that the integral exists for all $m$, $n$. Making the change of variable $u = x/(1-x)$ and $d\beta(x) = d\psi(x)/(1-x)^{2p}$, we see that the integral in the right-hand side is equal to

$$\int_0^{(v-1)^{-1}} u^{n+m}\, d\beta(u/(1+u)),$$

where $\beta(u/(1+u))$ is monotone increasing. Hence these are moments, $\beta_{n+m}$. In this case

$$(f, g)_c = \sum_{n,m=C}^{\infty} (-1)^{m+n} \binom{p+n-1}{n} \binom{p+m-1}{m} \beta_{n+m} f(n)\, g(m).$$

On the other hand, if $\phi(x) = e^{-x}$ and $d\psi(x)$ has $(0, \infty)$ for its support we have

$$\mu_{m,n} = (1/n!m!) \int_0^{\infty} x^{n+m} e^{-2x}\, d\psi(x),$$

which exists for all $m, n \geqslant 0$.

(a) *The Rice polynomials.* Let us take $\{\hat{P}_j(x),\ j = 0, 1, 2,\ldots\}$ to be the Legendre polynomial set with argument $1 - 2vx$, that is, in the hypergeometric notation,

$$\hat{P}_j(x) = {}_2F_1(-j; j+1; 1; vx),$$

so that $d\psi(x) = dx$ in $(0, 1/v)$ and $d\psi(x) = 0$ otherwise [7, Vol. 2], and take $\phi(x) = (1-x)^{-p}$, where $p$ is not a negative integer or zero.

From (1.4) we have

$$\hat{P}_j(x) = L[{}_3F_2[-j, j+1, -n; 1, p; v]; (1-x)^{-p}; x].$$

We recall that the Rice polynomials [12, 13] are defined by

$$H_j(\xi, p, v) = {}_3F_2(-j, j+1, \xi; 1, p; v). \tag{2.6}$$

Thus (2.3) is

$$\sum_{m,n=0}^{\infty} H_j(-n, p, v)\, H_l(-m, p, v) \binom{p+n-1}{n} \binom{p+m-1}{m} \eta_{m+n} = \frac{1}{v(2j+1)}\, \delta_{lj}, \tag{2.7}$$

where

$$\eta_k = (-1)^k \int_0^{1/v} x^k (1-x)^{-k-2p}\, dx.$$

It is known that the Rice polynomials $H_j(\xi, p, v)$ are not orthogonal in $v$ unless $\xi = 1$ or $\xi = p$. This follows from [1, Sect. 3]. If they are orthogonal in $\xi$ then $v = 1$ and $p$ is a negative integer. The "if" part follows from [1, Sect. 6] and the "only if" part from our Theorem 4.2 given below.

Thus these polynomials give an example of a set orthogonal with respect to discrete convolution but not orthogonal in the ordinary sense.

(b) *The Meixner polynomials.* These are defined [7, Vol. 2, p. 225] by means of

$$M_j(x, \beta, c) = {}_2F_1(-j, -x; \beta; 1 - c^{-1}).$$

According to (1.4) they are related to the Laguerre polynomials

$$L_n^{(\beta)}(x) = ((\beta+1)_j/j!)\, {}_1F_1(-j; \beta+1; x)$$

by means of

$$L[M_j(n; \beta, c); e^{-x}; x] = (j!/(\beta+1)_j)\, L_j(x(c^{-1}-1)). \tag{2.8}$$

The orthogonality relation for $L_j^{(\beta)}(bx)$ is [12, p. 206]

$$\int_0^{\infty} e^{-bx} x^{\beta} L_j^{(\beta)}(bx)\, L_l^{(\beta)}(bx)\, dx = (b^{-(\beta+1)}/j!)\, \Gamma(\beta+j+1)\, \delta_{lj}$$

where $R\beta > -1$. Thus we get from (2.3) and (2.4)

$$\sum_{r,s=0}^{\infty} M_j(r; \beta, c)\, M_l(s; \beta, c) \frac{c^{r+s}(\beta)_{r+s}}{(1+c)^{r+s}\, r!s!} = \frac{j!}{(\beta)_j} \left(\frac{1+c}{1-c}\right)^{\beta} \delta_{lj}. \tag{2.9}$$

This orthogonality relation is clearly different from the ordinary orthogonality relation [7, p. 225]

$$\sum_{n=0}^{\infty} M_j(n; \beta, c)\, M_l(n; \beta, c)\, \frac{c^n(\beta)_n}{n!} = j!\, \frac{c^{-j}(1-c)^{-\beta}}{(\beta)_j}\, \delta_{j,l}\,. \tag{2.10}$$

(c) *The Charlier polynomials.* These are given [7, Vol. 2, p. 226] by

$$c_j(x, a) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \binom{x}{k} k!a^{-k} = {}_2F_0(-j, -x; -, -1/a).$$

It is connected with the simple Laguerre polynomials $L_j(x) = {}_1F_1(-j; 1; x)$ by means of

$$L[c_j(n, a); J_0(2x^{1/2}); x] = L_j(x/a).$$

Therefore as above we get

$$\sum_{n,m=0}^{\infty} c_j(n, a)\, c_l(m, a)\, \theta_{m,n} = \delta_{lj}\,, \tag{2.11}$$

where

$$\theta_{m,n} = \frac{a^{m+n}}{m!n!} \sum_{r=0}^{\infty} \frac{(-a)^r}{r!} \binom{m+n+2r}{m+r}.$$

Again the orthogonality relation (2.11) is different from the known orthogonality relation

$$\sum_{r=0}^{\infty} c_m(r, a)\, c_n(r, a)\, (e^{-a}a^r/r!) = a^{-n}n!\delta_{nm}\,. \tag{2.12}$$

## 3. Connection Relations for Orthogonal Convolution Orthogonal Polynomials

Let $\{P_n(x)\}_0^\infty$ be a polynomial set satisfying the orthogonality relation

$$\lambda(j) \sum_{n=0}^{\infty} P_j(n)\, P_l(n)\, w(n) = \delta_{lj}\,. \tag{3.1}$$

and the convolution orthogonality relation

$$u(j) \sum_{m,n=0}^{\infty} \xi(m, n)\, P_j(m)\, P_l(n) = \delta_{lj}\,. \tag{3.2}$$

The uniqueness of the orthogonal polynomials satisfying (3.1) implies the connection relation

$$P_j(n) = \frac{u(j)}{\lambda(j)\, w(n)} \sum_{m=0}^{\infty} \xi(m, n)\, P_j(m). \tag{3.3}$$

These connection relations are very useful. One can use them to derive among other things, bilinear relations for the polynomials $\{P_n(x)\}_0^\infty$. We know at least one bilinear relation for the polynomials under consideration, namely, the dual orthogonality relation (see [6]), when the system of polynomials is complete:

$$w(k) \sum_{r=0}^{\infty} P_r(k)\, P_r(l)\, \lambda(r) = \delta_{kl}\,. \tag{3.4}$$

Multiply (3.3) by $\{\lambda(j)\}^2 P_j(s)/u(j)$ and sum over all nonnegative integers $j$. In view of (3.4) the result will reduce to

$$\sum_{j=0}^{\infty} \frac{\{\lambda(j)\}^2}{u(j)} P_j(s)\, P_j(n) = \frac{\xi(s, n)}{w(n)\, w(s)}\,. \tag{3.5}$$

One can repeat the above procedure by starting with (3.5) instead of (3.4). This leads in general to another bilinear relation, namely,

$$\sum_{j=0}^{\infty} \frac{\{\lambda(j)\}^3}{\{u(j)\}^2} P_j(s)\, P_j(n) = \frac{1}{w(n)} \sum_{m=0}^{\infty} \frac{\xi(s, m)}{w(s)} \frac{\xi(m, n)}{w(m)}\,. \tag{3.6}$$

Of course we can keep repeating this process but it is clear that very soon the sums will become complicated. On the other hand we can start with any bilinear relation other than (3.4).

To illustrate the above method let us apply it to the Charlier and Meixner polynomials.

For the Meixner polynomials we have

$$w(n) = \frac{c^n(\beta)_n}{n!}, \qquad \lambda(n) = \frac{c^n(\beta)_n}{n!}(1-c)^\beta, \qquad \xi(m, n) = \left(\frac{c}{1+c}\right)^{m+n} \frac{(\beta)_{m+n}}{m!n!},$$

$$u(n) = \frac{(\beta)_n}{n!}\left(\frac{1-c}{1+c}\right)^\beta.$$

Therefore the connection relation (3.3) is

$$M_j(x; \beta, c) = \sum_{m=0}^{\infty} \frac{(\beta + x)_m}{m!} c^{m-j}(1+c)^{-x-\beta-m} M_j(m; \beta, c) \tag{3.7}$$

and the bilinear relation (3.5) is

$$\sum_{j=0}^{\infty} \frac{c^{2j}(\beta)_j}{j!} M_j(x; \beta, c) M_j(y; \beta, c) = (1 - c^2)^{-\beta} (1 + c)^{-x-y} \frac{(\beta)_{x+y}}{(\beta)_x (\beta)_y}. \tag{3.8}$$

The bilinear relation (3.6) takes the simple form

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{c^{3j}(\beta)_j}{j!} M_j(x; \beta, c) M_j(y; \beta, c) \\ &\quad = (1 - c)^{-\beta} (1 + c)^{-2\beta-x-y} {}_2F_1\left(x + \beta, y + \beta; \beta; \frac{c}{(1 + c)^2}\right). \end{aligned} \tag{3.9}$$

We now treat the Charlier polynomials. The quantities $\lambda$, $w$, $u$, and $\xi$ are given by

$$w(n) = \frac{a^n}{n!}, \qquad \lambda(n) = e^{-a} \frac{a^n}{n!}, \qquad u(n) = 1,$$

$$\xi(m, n) = \frac{a^{m+n}}{m!n!} \sum_{r=0}^{\infty} \frac{(-a)^r}{r!} \binom{m + n + 2r}{m + r}.$$

The connection relation is given by

$$\begin{aligned} c_j(x, a) &= j! e^a a^{-j} \sum_{m=0}^{\infty} \frac{(x + 1)_m a^m}{(m!)^2} c_j(m, a) \\ &\quad \times {}_2F_2\left(\frac{x + m}{2}, \frac{x + m + 1}{2}; x + 1, m + 1; -4a\right). \end{aligned} \tag{3.10}$$

From (3.5) we obtain

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{a^{2j}}{j!\, j!} c_j(x, a) c_j(y, a) \\ &\quad = e^{2a} \binom{x + y}{x} {}_2F_2\left(\frac{x + y}{2}, \frac{x + y + 1}{2}; x + 1, y + 1; -4a\right). \end{aligned} \tag{3.11}$$

The connection relations (3.7) and (3.8) can be proved directly evaluating their respective left-hand sides and using Euler and Kummer's formulas

$${}_2F_1(a, b; c; z) = (1 - z)^{-a} {}_2F_1(a, c - b; c; -z/(1 - z)),$$

$${}_1F_1(a; b; z) = e^z {}_1F_1(b - a; b; -z).$$

Formulas (3.8) and (3.9) are special cases of the formula (see [7, Vol. 1, formula 12, p. 85).

$$\sum_{n=0}^{\infty} \binom{\lambda}{n} s^n \, {}_2F_1(-n, b; -\lambda; z) \, {}_2F_1(-n, \beta; -\lambda, \xi)$$

$$= (1+s)^{\lambda+b+\beta} (1+s-sz)^{-b} (1+s-s\xi)^{-\beta}$$

$$\times \, {}_2F_1\left(b, \beta; -\lambda; \frac{-z\xi s}{(1+s-sz)(1+s-s\xi)}\right),$$

We have not been able to find (3.10) or (3.11) in the standard tables and they are probably new.

It is of interest to find continuous analogs of the results in the present section. This remains as an open problem.

## 4. Characterizations

Although, in general, polynomials orthogonal with respect to convolution are not necessarily orthogonal in the ordinary sense (the sense of (1.5)) there are, however, polynomial sets such as the Charlier and the Meixner polynomials which are also orthogonal in the sense of (1.5). Furthermore according to our definition of convolution orthogonality every such set $\{Q_j(x)\}$ is a preimage under some $(L, \alpha)$ transform of an orthogonal polynomial set $\{P_j(x)\}$. Hence if we fix the set $\{P_j(x)\}$ we ask the question for what $(L, \alpha)$ transforms is the set $\{Q_j(x)\}$ also orthogonal. The answer to this question when $\{P_j(x)\}$ is either the set of Jacobi or the set of Laguerre polynomial is given in Theorems 3.1 and 3.2 respectively. In the present section we use "orthogonality" to mean orthogonality with respect to a finite signed measure instead of the positive measure as in the "ordinary" orthogonality.

THEOREM 4.1. *The only orthogonal polynomial sets that are preimages of Jacobi polynomials of argument* $1 - 2vx$ *under* $(L, \alpha)$ *transforms are* ${}_3F_2[-n, n+\gamma; x; \beta_1, \beta_2, 1]$.

*Proof.* Let $\{Q_j(x); j = 0, 1, 2,...\}$ *be such a set, so that*

$$((1+\alpha)_j/j!) \, {}_2F_1(-j, j+\alpha+\beta+1; \beta+1; vx) = L\{Q_j(n); \phi(x); x\}.$$

Hence $Q_j(x)$ is a polynomial of the form

$$Q_j(x) = \sum_{k=0}^{j} ((-j)_k/k!) (j+\gamma)_k (x)_k \lambda_k \qquad (\lambda_k \neq 0 \text{ for all } k). \tag{4.1}$$

Since these are assumed orthogonal it follows that they satisfy a three-term relation [14]

$$Q_{n+1}(x) = (A_n x + B_n) Q_n(x) + C_n Q_{n-1}(x) \qquad (A_n C_n \neq 0). \tag{4.2}$$

Substituting (4.1) in (4.2) and equating coefficients of $(x)_k$ we obtain

$$\begin{aligned}(n+1)(n+\gamma+k)&(n+\gamma+k-1)\\ &= -A_n(n+\gamma)\cdot(\lambda_{k-1}/\lambda_k)\,k - A_n k(n+\gamma)(n+\gamma+k-1)(n-k+1)\\ &\quad + B_n(n-k+1)(n+\gamma+k-1)(n+\gamma)\\ &\quad + (C_n/n)(n+\gamma)(n+\gamma-1)(n-k)(n-k+1).\end{aligned} \tag{4.3}$$

We note that $B_n + C_n = 1$. This follows by putting $x = 0$ in (4.2). Substituting for $B_n = 1 - C_n$ in (3.3) we get

$$\begin{aligned}(2n+\gamma+1)&(n+\gamma+k-1)\\ &= -(n+\gamma)\,A_n[(\lambda_{k-1}/\lambda_k) + (n+\gamma+k-1)(n-k+1)]\\ &\quad - (c_n/n)(2n+\gamma-1)(n-k+1)(n+\gamma)\end{aligned} \tag{4.4}$$

valid for $k = 1, 2,..., n+1$. Now to find $A_n$ put $k = n+1$ and after that put $k = 1$ to find the value of $C_n$. Having replaced the values of $A_n$ and $C_n$ in (4.4) we next put $k = n$ to get

$$\frac{\lambda_n}{n\lambda_{n+1}} - \frac{\lambda_{n-1}}{(n-1)\lambda_n} = 1 - \frac{\lambda_0}{\lambda_1}\frac{1}{n(n-1)}.$$

The general solution of this first-order recurrence relation (in $(\lambda_n/n\lambda_{n+1})$) is

$$\frac{\lambda_{n+1}}{\lambda_n} = \frac{1}{n^2 + bn + c} = \frac{1}{(n+\beta_1)(n+\beta_2)}$$

so that

$$\lambda_n = \frac{1}{(\beta_1)_n (\beta_2)_n}.$$

This completes the proof of the theorem.

THEOREM 4.2. *The only orthogonal polynomial sets that are preimages of Laguerre polynomials under $(L, \alpha)$ transforms are the Meixner and Charlier polynomials.*

The proof of Theorem 4.2 is very similar to that of Theorem 4.1 and uses Theorem 3 of [1]. It is clear that Theorems 4.1 and 4.2 are valid if "Jacobi polynomials of argument $1 - 2vx$" and "Laguerre polynomials" are replaced by polynomials of the type $\sum_{k=0}^{n} (-n)_k (n+\gamma)_k \xi_k x^k$ and $\sum_{k=0}^{n} (-n)_k \xi_k x^k$, respectively, with $\xi_k \neq 0$ for all $k$.

Note that Theorems 4.1 and 4.2 are also extensions of Theorems 4 and 3 of [1] respectively.

For preimages of symmetric polynomials under $(L, \alpha)$ transforms we have

THEOREM 4.3. *There are no orthogonal polynomials that are preimages of symmetric polynomials under any $(L, \alpha)$ transform.*

*Proof.* If $\{Q_n(x)\}_0^\infty$ is such a set, then

$$Q_n(x) = \sum_{k=0}^{[n/2]} \lambda_{n,k}(x)_{n-2k},$$

and the result follows from the three-term recurrence relation, after some manipulations.

It is clear that if we specify an orthogonal polynomial set $\{P_n(x)\}_0^\infty$, then the problem of determining orthogonal preimages under $(L, \alpha)$ transform can be treated as in the above cases. However the problem of determining all pairs of orthogonal polynomials $(\{P_n(x)\}_0^\infty, \{Q_n(x)\}_0^\infty)$ such that $Q_n(x)$ is preimage of $P_n(x)$ under an $(L, \alpha)$ transform for all $n$, seems to be much more general and remains to be investigated.

As a problem in the reverse direction we give the following theorem.

THEOREM 4.4. *The only orthogonal polynomial sets which are $(L, \alpha)$ images of polynomials of type*

$$Q_j(x) = \sum_{k=0}^{j} \frac{(-j)_k (-x)_k}{k!} \xi_k \qquad (x = 0, 1, 2, \ldots), \tag{4.5}$$

*where $\xi_0 = 1$, $\xi_n \neq 0$ for all $k$ is the Laguerre polynomial set.*

*Proof.* Under $(L, \alpha)$ transform $\{Q_j(x)\}$ goes into (putting $\eta_k = (-1)^k k! \xi_k \alpha_k$)

$$P_j(x) = \sum_{k=0}^{j} (-1)^k \frac{(-j)_k \eta_k}{k!} x^k.$$

Since this is assumed to be orthogonal then it satisfies a three-term recurrence relation

$$P_{n+1}(x) = \left(\frac{\eta_{n+1}}{\eta_n} x + B_n\right) P_n(x) + C_n P_{n-1}(x) \qquad (n \geqslant 0).$$

$$P_{-1}(x) = 0, \qquad P_0(x) = 1.$$

Equating coefficients of $x^k$ we get after some simplication

$$n(n+1) = nk \frac{\eta_{n+1}}{\eta_n} \frac{\eta_{k-1}}{\eta_k} + B_n n(n+1-k) + C_n(n-k)(n+1-k)$$

$$0 \leqslant k \leqslant n+1. \qquad (4.6)$$

The value $k = 0$ shows that $C_n + B_n = 1$, $(n \geqslant 1)$. Hence substituting this value in (4.6) we get

$$n = n \frac{\eta_{n+1}}{\eta_n} \frac{\eta_{k-1}}{\eta_k} - (n+1-k)\, C_n\,,$$

which implies (by putting $k = 1$) then $C_n = (\eta_{n+1}/\eta_n)(\eta_0/\eta_1) - 1$. Thus

$$\frac{\eta_n}{n\eta_{n+1}} - \frac{\eta_{n-1}}{(n-1)\,\eta_n} = -\frac{\eta_0}{\eta_1} \frac{1}{n(n-1)}\,,$$

whose general solution is

$$\eta_n/\eta_{n+1} = (\eta_0/\eta_1) + cn,$$

which in turn yields that after a slight change of notation

$$\eta_n = \alpha^n/(\beta)_n$$

where

$$\eta_1 = \alpha/\beta.$$

This results show that $P_j(x) = L_j^{(\beta-1)}(-\alpha x)$ which was to be proved.

## 5. Zeros of Certain Polynomials Orthogonal with Respect to Discrete Convolution

Here we restrict ourselves to sequences $\{f(n)\}$ for which $\sum_{n=0}^{\infty} (x^n/n!)\, f(n)$ is bounded on $(0, \infty)$ and $\phi(x)$ which is a Laplace transform of a integrable (on $(0, \infty)$) and nonnegative function $w(t)$, i.e.,

$$\phi(x) = \int_0^\infty e^{-xt} w(t)\, dt, \qquad w(t) \geqslant 0 \quad \text{on} \quad (0, \infty), \qquad (5.1)$$

where

$$1 = \int_0^\infty w(t)\, dt.$$

In this case the $(L, \phi)$ transform can be written as

$$L[f; \phi; x] = \int_0^\infty e^{-xt} \sum_{n=0}^{\infty} ((xt)^n/n!) f(n) w(t) \, dt. \tag{5.2}$$

An operator $T$ is variation diminishing if $V[Tf] \leqslant V[f]$, where $V[f]$ is the variaiton of $f$ in the interval in question (for definition see [7, p. 83]). We shall need the following lemmas.

LEMMA 1. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *be uniformly convergent on* $[0, a]$ *then on this interval* $V[f] \leqslant V[\{a_k\}]$, *where* $V[\{a_k\}]$ *is the number of sign change in the sequence* $a_0, a_1, a_2, \ldots$.

LEMMA 2. *If* $f(x) = \int_0^\infty e^{-xt} g(t) \, dt$, *then on* $(0, \infty)$ $V[f] \leqslant V[g]$.

Lemma 1 is an easy consequence of Descartes' rule of signs and was first observed by Cheney and Sharma [5], whereas Lemma 2 follows from [8, Theorem 7.1, p. 97, and Theorem 9.1a, p. 103].

We now give the following theorem.

THEOREM 5.1. *If* $\{P_j(x), j = 0, 1, \ldots\}$ *is a polynomial set orthogonal* (*in the ordinary sense*) *on* $(0, \infty)$ *and if* $Q_j(n) = L^{-1}[P_j(x); \phi; n]$, *where* $\phi(t)$ *is of the class* (4.1), *then* $Q_j(x)$ *has real and simple zeros in the interval* $(0, \infty)$.

*Proof.* The relationship between $\{P_j(x)\}$ and $\{Q_j(x)\}$ can be written as

$$P_j(x) = \int_0^\infty e^{-xt} \left\{ \sum_{n=0}^{\infty} ((xt)^n/n!) Q_j(n) \right\} w(t) \, dt \qquad (j = 0, 1, 2, \ldots).$$

Hence from Lemma 2 and for each $j$ we have on the interval $(0, \infty)$

$$V[P_j(x)] \leqslant V \left\{ \sum_{n=0}^{\infty} ((xt)^n/n!) Q_j(n) \right\}$$

and from Lemma 1 we get

$$V[P_j(x)] \leqslant V \left\{ \sum_{n=0}^{\infty} ((xt)^n/n!) Q_j(n) \right\} \leqslant V[\{Q_j(n)\}].$$

Now since $\{P_j(x)\}\}$ is an orthogonal set in $(0, \infty)$ we know that all its zeros are real, simple and lie in $(0, \infty)$. Hence $V[P_j(x)] = j$. Hence

$$V[\{Q_j(n)\}] \geqslant j.$$

Since $Q_j(x)$ is a polynomial in $x$ of degree exactly $j$ it follows that $V[\{Q_j(n)\}] \leqslant j$. Combining these two inequalities we get that $V[\{Q_j(n)\}] = j$. This proves the theorem.

COROLLARY. *The zeros of the Rice polynomials* (2.8) *are real and simple for positive $p$ and negative $v$.*

*Proof.* It is easy to see that

$$\tilde{P}_j(x) = L[H_j(-n, p, -v); (1 + x)^{-p}; x],$$

where $\tilde{P}_j(x)$ is the shifted Legendre polynomial which is orthogonal on $(0, 1/v)$. In this case $\phi(x)$ is the Laplace transform of $t^{p-1}e^{-t}/\Gamma\Gamma(p)$. Hence the polynomial (in $x$) $H_j(x, p, -v)$ has real, simple and nonpositive zeros for $p > 0$ and $v > 0$.

The above result does not hold for positive $p$ and $v$. For example, $H_2(x, p, p + 1)$ has complex zeros.

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